ON THE CLOSURE OF THE DIAGONAL OF A T_1-SPACE

MARIA LUISA COLASANTE AND DOMINIC VAN DER ZYPEN

ABSTRACT. Let X be a topological space. The closure of $\Delta = \{(x,x) : x \in X\}$ in $X \times X$ is a symmetric relation on X. We characterise those equivalence relations on an infinite set that arise as the closure of the diagonal with respect to a T_1- topology.

1. INTRODUCTION

Our starting point is the following well-known proposition. A topological space (X, τ) is T_2 if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is a closed subset of $X \times X$.

Generally speaking, the closure of the diagonal is a reflexive relation on X. That relation can be used to characterise further separation axioms on topological spaces. This has been done in [3].

We use the following simple observation throughout the paper:

> Let (X, τ) be a topological space and let $x, y \in X$. Then $(x,y) \notin Cl_\tau(\Delta)$ if and only if x and y have disjoint open neighborhoods.

This statement is straightforward to prove; moreover it implies that $Cl_\tau(\Delta)$ is always a symmetric relation. We make use of the observation without explicitly mentioning it.

As a simple example, consider any infinite set X and equip it with the topology $\mathcal{P}_{cf}(X)$, ie. the collection of all cofinite sets together with the empty set. Note that a topology τ is T_1 iff $\tau \supseteq \mathcal{P}_{cf}(X)$. In $\mathcal{P}_{cf}(X)$ any two members intersect. So by the observation made above one obtains $Cl_\tau(\Delta) = X \times X$.

Moreover note that for two topologies $\sigma \supseteq \tau$ one obtains $Cl_\sigma(\Delta) \subseteq Cl_\tau(\Delta)$. Which reflexive and symmetric relations on a set X can be represented as the closure of the diagonal on $X \times X$ with respect to some topology? This question is very natural and is still open. We confine ourselves to equivalence relations and T_1-spaces.

Definition 1.1. *Let R be an equivalence relation on an infinite set X. Then R is said to be T_1-realisable if there is a T_1 topology on X such that $R = Cl_\tau(\Delta)$.*

The main goal of this article is to characterise T_1-realisable equivalence relations and we provide a characterisation in theorem 3.7. Moreover, we show in Example 4.1 that the closure of the diagonal need not be transitive.

The following tool is straightforward to prove. It allows us to consider just *bases* instead of whole topologies when dealing with the closure of the diagonal. By a basis we mean a collection $\mathcal{B}$ of open subsets of a set X, such that for all $B_1, B_2 \in \mathcal{B}$ one obtains

> for every $x \in B_1 \cap B_2$ there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq B_1 \cap B_2$.

Lemma 1.2. *Let X be a topological space and let $\mathcal{B}$ be a basis of X. Then $(x, y) \notin Cl_\tau(\Delta)$ if and only if there exist disjoint members $A, B \in \mathcal{B}$ such that $x \in A$ and $y \in B$.*

If we consider all topologies on a set X, then every equivalence relation can be written as the closure of the diagonal of some topology, as the following shows:

Let R be an equivalence relation on X For each $x \in X$, we let $R(x) = \{y \in X : (x, y) \in R\}$ denote the equivalence class of x. Let

$$\tau_R = \{V \subset X : x \in V \text{ implies } R(x) \subset V\}.$$

It is easy to see that $\{R(x) : x \in X\}$ is a basis for the topology and that therefore x, y can be separated by disjoint open sets iff $(x, y) \notin R$.

Next we introduce some notation. It is well known that equivalence relations and partitions of a set X are in a natural correspondence. For an equivalence relation R let $\text{Part}(R) = \{R(x) : x \in X\}$ and each partition $\mathcal{P}$ on X let $\text{Eq}(\mathcal{P}) = \{(x, y) \in X \times X : (\exists B \in \mathcal{P}) : x, y \in P\}$. It is obvious that $\text{Part}(R)$ is a partition and $\text{Eq}(\mathcal{P})$ is an equivalence relation on X. The elements of a partition are called "blocks". We also refer to the the equivalence classes of an equivalence relation R (ie the blocks of $\text{Part}(R)$) as "blocks".

2. Infinite blocks only

We present a solution for the following particular case: R is an equivalence relation on a set X such that $R(x)$ is infinite for each $x \in X$.

Proposition 2.1. *If R is an equivalence relation on an infinite set X such that every block of R is infinite, then R is T_1-realisable.*

Proof. Consider the following topology on X:

$$\tau = \{X, \emptyset\} \cup \{V \subset X : x \in V \text{ implies } R(x) \cup V \text{ is cofinite in} R(x)\}$$

It is easy to check that τ is T_1. Suppose that $(x, y) \in R$ and pick any open neighborhoods U, V of x and y, respectively. Clearly, $U \cap R(x)$ and $V \cap R(x)$ are cofinite subsets of the infinite set $R(x)$ by definition of τ, so they intersect. Conversely suppose $(x, y) \notin R$. Then $R(x)$ and $R(y)$ are disjoint open neighborhoods of x and y, respectively. So one concludes that $R = Cl_\tau(\Delta)$. $\qquad\square$

3. Allowing for finite blocks

For any equivalence relation R on a set X we define three important sets: $\mathcal{C}_{\{*\}}(R)$, the set of all singleton blocks, $\mathcal{C}_{\mathrm{fin}>1}(R)$, the set of all finite blocks containing more than one element, and $\mathcal{C}_\infty(R)$, the set of all infinite blocks.

More formally, we let $\mathcal{C}_{\{*\}}(R) = \{B \in \mathrm{Part}(R) : B = \{x\} \text{ for some } x \in X\}$ and $\mathcal{C}_\infty(R) = \{B \in \mathrm{Part}(R) : B \text{ is infinite}\}$ and last $\mathcal{C}_{\mathrm{fin}>1}(R) = \mathrm{Part}(R) \setminus (\mathcal{C}_{\{*\}}(R) \cup \mathcal{C}_\infty(R))$.

Proposition 3.1. *If all blocks of an equivalence relation R are infinite or singletons, R is T_1-realisable.*

Proof. We just give a sketch. Let $Y = \bigcup \mathcal{C}_\infty(R)$ and let $Z = \bigcup \mathcal{C}_{\{*\}}(R)$. Clearly, the restriction of R to Y has only infinite blocks. Endow Y with the topology described in proposition 2.1 and give Z the discrete topology. It is easy to see that this is a basis for a topology on $X = Y \cup Z$ such that $R = Cl_\tau(\Delta)$. $\qquad\square$

If we allow for finite blocks with more than one element, considerations get more involved.

Proposition 3.2. *Let R be an equivalence relation on an infinite set X such that $\mathcal{C}_{\mathrm{fin}>1}(R)$ is finite and nonempty.*

(1) *If $\mathrm{Part}(R)$ is finite then R is not T_1-realisable.*
(2) *If $\mathrm{Part}(R)$ is infinite then R is T_1-realisable.*

Proof. (1) Assume that there is a T_1 topology τ such that $Cl_\tau(\Delta) = R$. Pick $a \in X$ such that $R(a)$ is finite and there exists $b \neq a$ such that $(a, b) \in R$.

Claim 1. Every open neighborhood of a is infinite. - If there were a finite neighborhood of a then $\{a\}$ would be open since τ is T_1. So $\{a\}$ and $X \setminus \{a\}$ separate a and b, which implies $(a, b) \notin Cl_\tau(\Delta) = R$, contradicting our choice of a, b.

Claim 2. There exists $B^* \in \mathcal{C}_\infty(R)$ such that for every open neighborhood U of a one obtains $B^* \cap U \neq \emptyset$. Suppose the contrary, so for all $B \in \mathcal{C}_\infty(R)$ there is an open neighborhood U_B of a such that $U_B \cap B = \emptyset$. Then $U' = \bigcap \{U_B : B \in \mathcal{C}_\infty(R)\}$ is open since $\mathcal{C}_\infty(R)$ is finite, and $U' \subseteq \bigcup \mathcal{C}_{\mathrm{fin}>1}(R) \cup \bigcup \mathcal{C}_{\{*\}}(R)$,

which is a finite set. So U' is a finite neighborhood of a, contradicting claim 1. So claim 2 is proved.

Now pick B^* from claim 2 and let $x \in B^*$. We show that x and a cannot be separated by open sets, which then contradicts $(a, x) \notin R$. - Pick open neighborhoods U, V of a and x, respectively. Using claim 2, pick $y \in B^* \cap U$. Since $(x, y) \in R$, they cannot be separated by open neighborhoods and since U is an open neighborhood of y, one obtains that U and V intersect, and we are done.

(2) We distinguish two cases:

Case 1. $\mathcal{C}_{\{*\}}(R)$ is infinite. Recall that $\mathcal{C}_{\mathrm{fin}>1}(R)$ is finite by assumption of the proposition. For each $C \in \mathcal{C}_{\mathrm{fin}>1}(R)$ pick an infinite subset $S_C \subseteq \bigcup \mathcal{C}_{\{*\}}(R)$ such that if $C, D \in \mathcal{C}_{\mathrm{fin}>1}(R)$ then $S_C \cap S_D = \emptyset$. We give a basis for a topology τ by

$$
\begin{aligned}
\mathcal{B} = \ & \{\{x\} : x \in \bigcup \mathcal{C}_{\{*\}}(R)\} \ \cup \\
& \{U \subseteq X : U \in \mathcal{P}_{\mathrm{cf}}(B) \text{ for some } B \in \mathcal{C}_\infty(R)\} \ \cup \\
& \{V \subseteq X : V \cap (\bigcup \mathcal{C}_{\mathrm{fin}>1}(R)) = \{x\} \text{ for some } x \text{ and } V = \{x\} \cup U \\
& \qquad \qquad \text{for some } U \in \mathcal{P}_{\mathrm{cf}}(S_{R(x)})\}.
\end{aligned}
$$

We argue shortly that $\mathcal{B}$ is indeed a basis. Designate the three "parts" of $\mathcal{B}$ by P_1, P_2, P_3 such that $\mathcal{B}$ is the disjoint union of P_1, P_2 and P_3. Clearly, the intersection of two members of P_i for $i = 1, 2$ is empty or again in P_i. Let $V, W \in P_3$ and suppose that $V \cap (\bigcup \mathcal{C}_{\mathrm{fin}>1}(R)) = \{x\}$ and $W \cap (\bigcup \mathcal{C}_{\mathrm{fin}>1}(R)) = \{y\}$. Moreover, let $V = \{x\} \cup U$ for some $U \in \mathcal{P}_{\mathrm{cf}}(S_{R(x)})$ and let $W = \{y\} \cup U'$ for some $U' \in \mathcal{P}_{\mathrm{cf}}(S_{R(y)})$. If $x \neq y$ then $V \cap W = \emptyset$. If $x = y$ then $U \cap U' \in \mathcal{P}_{\mathrm{cf}}(S_{R(x)})$ and $V \cap W = \{x\} \cup (U \cap U')$. So $V \cap W \in \mathcal{B}$. Last, let $M_i \in P_i$. We get $M_1 \cap M_2 = M_2 \cap M_3 = \emptyset$ and $M_1 \cap M_3$ is empty or contains for each $x \in M_1 \cap M_3$ a member B of $\mathcal{B}$ such that $x \in B \subseteq M_1 \cap M_3$: take $B = \{x\}$.

Moreover, a case distinction shows that $x, y \in X$ can be separated by disjoint members of $\mathcal{B}$ if and only if $(x, y) \notin R$. And for $x \neq y$ in X there are basic open sets containing x but not y and vice versa. So the topology generated by $\mathcal{B}$ is T_1.

Case 2. $\mathcal{C}_{\{*\}}(R)$ is finite, so $\mathcal{C}_\infty(R)$ is infinite since $\mathrm{Part}(R)$ is infinite. Recall that $\mathcal{C}_{\mathrm{fin}>1}(R)$ is finite by assumption of the proposition. For each $C \in \mathcal{C}_{\mathrm{fin}>1}(R)$ pick an infinite subset $\mathcal{S}_C \subseteq \mathcal{C}_\infty(R)$ such that if $C, D \in \mathcal{C}_{\mathrm{fin}>1}(R)$ then $\mathcal{S}_C \cap \mathcal{S}_D = \emptyset$. Note that there is a subtle difference in the definition of S_C in the above case and $\mathcal{S}_C$ here: in case 1, S_C was a subset of X, and here $\mathcal{S}_C$ is a subset of $\mathcal{C}_\infty(R)$. We give a basis for a topology τ by

$$
\begin{aligned}
\mathcal{B} = \ & \{\{x\} : x \in \bigcup \mathcal{C}_{\{*\}}(R)\} \ \cup \\
& \{U \subseteq X : U \in \mathcal{P}_{\mathrm{cf}}(B) \text{ for some } B \in \mathcal{C}_\infty(R)\} \ \cup \\
& \{V \subseteq X : V \cap (\bigcup \mathcal{C}_{\mathrm{fin}>1}(R)) = \{x\} \text{ for some } x \text{ and } V = \{x\} \cup (\bigcup \mathcal{T})
\end{aligned}
$$

$$\text{for some } \mathcal{T} \in \mathcal{P}_{\text{cf}}(\mathcal{S}_{R(x)})\}.$$

In a very similar way to what we did in case 1, we can check that $\mathcal{B}$ is a basis. And again, the topology generated by $\mathcal{B}$ is T_1. Moreover, a case distinction shows that $x, y \in X$ can be separated by disjoint members of $\mathcal{B}$ if and only if $(x, y) \notin R$. □

Corollary 3.3. *On an infinite set, an equivalence relation with finitely many blocks is T_1-realisable if and only if every finite block is a singleton.*

Proposition 3.2 describes what happens if $\mathcal{C}_{\text{fin}>1}(R)$ is finite and nonempty. Next we we look at what happens if $\mathcal{C}_{\text{fin}>1}(R)$ is infinite. First we prove that if each block consists of exactly two points, then the relation is T_1-realisable.

Lemma 3.4. *Let X be an infinite set and R be an equivalence relation such that $\mathrm{card}(R(x)) = 2$ for all $x \in X$. Then R is T_1-realisable.*

Proof. We construct a topological space (Z, σ) whose ground set Z is equinumerous to X and equip it with an equivalence relation S such that $\mathrm{card}(S(z)) = 2$ for all $z \in Z$ such that the above-mentioned properties are satisfied. It is not difficult to prove that there is a bijection $\varphi : Z \to X$ such that $(z_1, z_2) \in S$ if and only if $(\varphi(z_1), \varphi(z_2)) \in R$. Then $\tau = \{\varphi(U) : U \in \sigma\}$ is a topology satisfying the condition of the lemma. Let $Z = X \times \mathbb{Q} \times \{0, 1\}$. Note that clearly Z is equinumerous with X. Moreover we let the equivalence relation S on Z be defined by

$$(x, q, k) \sim_S (x', q', k') \text{ iff } x = x' \text{ and } q = q'.$$

Clearly, each block of S has 2 elements.

We define $B \subseteq Z$ as basic open if and only if there are $x \in X, q \in \mathbb{Q}$ and $\delta \in \mathbb{Q}_{>0}$ such that B is a cofinite subset of

$$B_\delta(x, q) := \{x\} \ \times \ B_\delta(q) \ \times \ \{0, 1\},$$

where $B_\delta(q) = \{q' \in \mathbb{Q} : |q' - q| < \delta\}$. Intuitively speaking, $B_\delta(x, q)$ is a series of 2 copies of $B_\delta(q)$, such that each copy lies at its appropriate place in the set $X \times \mathbb{Q} \times \{k\}$ for each k.

It is easy to verify that the collection of all basic open sets is a basis that gives rise to a T_1-topology. Moreover the following is readily verified: members (x, q, k) and (x', q', k') of Z can be separated by basic open sets if and only if $x \neq x'$ or $q \neq q'$. So using lemma 1.2 and the reasoning at the beginning of this proof we are done. □

Proposition 3.5. *Let X be an infinite set and R an equivalence relation on X. Suppose that $\mathrm{Part}(R)$ is infinite and each block has more than one element. Then R is T_1-realisable.*

Proof. For each $B \in \mathrm{Part}(R)$ pick two distinct representatives $r_1(B), r_2(B) \in B$. Let $W = \{r_1(B) : B \in \mathrm{Part}(R)\} \cup \{r_2(B) : B \in \mathrm{Part}(R)\}$ and let

$S = R \cap (W \times W)$. Clearly, each block of S has just two elements, so by lemma 3.4 there is a T_1-topology σ on W such that $Cl_\sigma(\Delta_W) = S$.

Using σ we equip X with a topology having the desired property. We say that $V \subseteq X$ is basic open if and only if one of the following two conditions holds:

(1) $V \in \sigma$, that is $V \subseteq W$ and V is open;
(2) there is $x \in X \setminus W$ and $A \in \sigma$ with $A \cup \{r_1(R(x))\} \in \sigma$ and $B = \{x\} \cup A$.

As we easily verify, the collection $\mathcal{B}$ of basic open elements is indeed a basis and very intuitively speaking "things in the basis happen more or less on W". We define τ to be the topology generated by $\mathcal{B}$.

Suppose that $(x, y) \in R$ and that V_x, V_y are basic open sets containing x and y, respectively. So this leads to case distinction. Suppose that $x, y \notin \{r_1(R(x)), r_2(R(x))\}$. Then there are A_x, A_y in σ such that $V_x = \{x\} \cup A_x$ and $V_y = \{y\} \cup A_y$ and $A_x \cup \{r_1(R(x))\} \in \sigma$ and $A_y \cup \{r_1(R(y))\} \in \sigma$ where of course $R(x) = R(y)$. So $(A_x \cap A_y) \cup \{r_1(R(x))\} \in \sigma$. Moreover, every neighborhood of $r_1(R(x))$ in (W, σ) is infinite. (Suppose otherwise: since σ is T_1, we could separate $r_1(R(x))$ and $r_2(R(x))$ by disjoint open sets in W, contradicting $(r_1(R(x)), r_2(R(x))) \in S = Cl_\sigma(\Delta_W)$.) So since $(A_x \cap A_y) \cup \{r_1(R(x))\}$ is an open neighborhood of $r_1(R(x))$, it can not just consist of $r_1(R(x))$, therefore A_x, A_y have nonempty intersection, and so have their respective supersets V_x and V_y. The other cases are treated in a similar way.

Suppose that conversely one obtains $(x, y) \notin R$. One has distinguish some cases. Assume that $x \notin \{r_1(R(x)), r_2(R(x))\}$ and $y \notin \{r_1(R(y)), r_2(R(y))\}$. Look at $r_1(R(x))$ and $r_1(R(y))$. Since $(r_1(R(x)), r_1(R(y))) \notin S = Cl_\sigma(\Delta_W)$, they can be separated by disjoint open neighborhoods U_1, U_2, respectively. By definition of $\mathcal{B}$, the sets $\{x\} \cup U_1$ and $\{y\} \cup U_2$ are disjoint basic open sets separating x and y. The remaining cases are treated in a similar way, implying that x and y can always be separated by basic open sets if $(x, y) \notin R$.

So we get $R = Cl_\tau(\Delta)$.   □

Corollary 3.6. *If R is an equivalence relation on a set X such that $\mathrm{Part}(R)$ is infinite, then R is T_1-realisable.*

Proof. The case that $\mathcal{C}_{\mathrm{fin}>1}(R)$ is finite has been dealt with. So suppose that $\mathcal{C}_{\mathrm{fin}>1}(R)$ is infinite. Note that X is the disjoint union of $A = \bigcup \mathcal{C}_{\{*\}}(R)$, $B = \bigcup \mathcal{C}_{\mathrm{fin}>1}(R)$ and $C = \bigcup \mathcal{C}_\infty(R)$. Endow B with the topology described in proposition 3.5 and give $A \cup C$ the topology constructed in 3.1. The disjoint union of the two topological spaces just mentioned give a T_1-topology on X such that $Cl_\tau(\Delta) = R$.   □

The above results can be summarised in the following theorem.

Theorem 3.7. *Suppose that X is an infinite set and R is an equivalence relation on X. Then the following are equivalent:*

(1) *R is not T_1-realisable;*
(2) *$\mathrm{Part}(R)$ is finite and R has a finite block that is not a singleton.*

4. FURTHER DIRECTIONS

First we construct a space such that the closure of the diagonal is not transitive.

Example 4.1. *On $X = \omega$ we let*

- *$D_1 = \{3n + 1 : n \in \omega\}$, and*
- *$D_2 = \{3n + 2 : n \in \omega\}$.*

Let τ be the topology on ω generated by the subbasis

$$\mathcal{S} = \mathcal{P}_{\mathrm{cf}}(\omega) \cup \{D_1\} \cup \{D_2\}.$$

Obviously, τ is T_1.

We obtain $(2,3) \in Cl_\tau(\Delta)$ and $(3,4) \in Cl_\tau(\Delta)$ but $(2,4) \notin Cl_\tau(\Delta)$. So $Cl_\tau(\Delta)$ is not transitive.

In section 1 we saw that any equivalence relation is realisable by some topology, although the topology given there is in general not even T_0.

Proposition 4.2. *Every equivalence relation R on a set X is T_0-realisable.*

Proof. Let R be an equivalence relation on the set X. For each block $B \in \mathrm{Part}(R)$ we pick a representative $r(B) \in B$ and we define $U \subseteq X$ to be open if and only if

   if $U \cap B \neq \emptyset$ for some $B \in \mathrm{Part}(R)$ then $r(B) \in U$.

It is not difficult to see that the collection of open sets form a topology. To see that this collection is T_0 pick $x \neq y \in X$. If $(x, y) \notin R$ then $R(x)$ is an open set that contains x but not y. If $(x, y) \in R$ then at least one of x, y does not equal $r(B)$ where $B = R(x) = R(y)$ and we may assume $y \neq r(B)$. Then $\{x, r(B)\}$ is an open set containing x but not y.

Next we show that $Cl_\tau(\Delta) = R$. Let $(x, y) \in (X \times X) \setminus R$. Then $R(x)$ and $R(y)$ are disjoint open sets containing x and y, respectively. So $(x, y) \notin Cl_\tau(\Delta)$. Conversely take $(x, y) \in R$. By construction of the open sets, every open neighborhood of x and every open neighborhood of y contains $r(R(x)) = r(R(y))$. So, x, y cannot be separated by disjoint open sets and implies that $(x, y) \in Cl_\tau(\Delta)$. $\qquad\square$

The present article characterises T_1-realisable equivalence relations. A natural extension of this is the examination of the realisability of symmetric and reflexive binary relations without the requirement that the relation be transitive. We want to conclude the article with two open questions:

Question 4.3. *Let X be any set. For which symmetric and reflexive relations R does there exist a topology on X such that $Cl_\tau(\Delta) = R$? What happens if we confine ourselves to T_0- or T_1-spaces?*

The more "natural" spaces tend to have a transitive closure of their diagonal. This issue is addressed in the following question:

Question 4.4. *Characterise those topological spaces such that the closure of the diagonal is transitive. Is there a "geometrical interpretation" of transitivity?*

5. Acknowledgements

References

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(M.L.Colasante) Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida 5101, Venezuela, `marucola@ula.ve`

(D.van der Zypen) Allianz Suisse Insurance Company, Laupenstrasse 27, CH-3001 Bern, Switzerland, `dominic.zypen@gmail.com`